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Effective Methods of Determining the Modulus of Doubly Connected Domains

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The modulus of a doubly connected domain is determined by a quotient of certain kernel functions, namely the Bergman kernel, the reduced Bergman kernel and the square of the Szegö kernel. These methods are more efficient than methods involving the curvature of the Bergman metric.

1. INTRODUCTION

Among the many methods for determining the modulus of a doubly connected domain (see for example the book of Gaier [4, pp. 180–257]) methods employing the classical theory of kernel functions seem to be very effective [3]. This is due to the relative simplicity of the numerical computation of the kernel functions. Usually, these methods involve (cf. [2, 3]) the Gaussian curvature of the Bergman metric induced by the Bergman kernel. In this paper we use quotients of certain kernel functions as a substitute for the conformal invariant curvature. These quantities are of course simpler and easier to compute. For example, the modulus of a doubly connected domain D can be computed at once by considering the ratio of the reduced Bergman kernel and the square of the Szegö kernel (Theorem 1).

In Section 2 we briefly review the classical theory of kernel functions and the main results are elaborated in Section 3. After Theorem 1 is proved, the numerical algorithm described in Section 4 proceeds along known lines exhibited in [3].

2. GENERAL THEORY

Let D be a bounded plane domain and to avoid unessential difficulties let us assume that its boundary ∂D is a sum of p nondegenerate analytic boundary components Γ_j, $0 \leqslant j \leqslant p-1$, where Γ_0 is the outer component. In the sequel it will be clear that this restriction can be relaxed in various directions. For instance, it is sufficient to assume that each Γ_k is of class $\mathscr{C}^2$ with respect to

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the arc length s on ∂D. The classical theory of kernel functions (Bergman [1, pp. 21, 49, 108]) allows to introduce the Bergman kernel function $K(z, \bar{\xi})$, the reduced Bergman kernel $\tilde{K}(z, \bar{\xi})$ and the Szegö kernel $\hat{K}(z, \bar{\xi})$ for D. These well-known kernels are uniquely defined as reproducing kernels for certain Hilbert spaces of analytic functions in D, which are described as follows.

Let $L_2(D)$ denote the Hilbert space of complex-valued functions on D which are square integrable with respect to the Lebesgue measure. We take the usual scalar product. Let $H(D)$ designate the class of all functions analytic in D. Define $L_2H(D) = L_2(D) \cap H(D)$ and let $L_2^{(s)}H(D) = \{f \in L_2H(D): f$ has a single-valued indefinite integral$\}$. Clearly, $L_2H(D)$ is a closed subspace of $L_2(D)$ and $L_2^{(s)}H(D)$ is a closed subspace of $L_2H(D)$. Moreover, $L_2H(D) = L_2^{(s)}H(D) \oplus E(D)$, where $E(D)$ is the orthogonal complement of $L_2^{(s)}H(D)$ in $L_2H(D)$. Here $\dim E(D) = p - 1$ and the basis for $E(D)$ are the functions $w_k' = \partial w_k/\partial z$, where $w_k(z)$ denotes the usual harmonic measure for Γ_k with respect to D, $1 \leqslant k \leqslant p - 1$. The Hilbert spaces $L_2H(D)$, $L_2^{(s)}H(D)$, and $E(D)$ possess the reproducing kernels $K(z, \bar{\xi})$, $\tilde{K}(z, \bar{\xi})$ and $E(z, \bar{\xi})$ respectively and so $K(z, \bar{\xi}) = \tilde{K}(z, \bar{\xi}) + E(z, \bar{\xi})$.

To define the Szegö kernel we have to replace $L_2(D)$ by $L_2(\partial D)$. This Hilbert space consists of those complex-valued functions which are normed by $\|f\| = (f, f)^{1/2}$, where $(f, g) = \int_{\partial D} f(z)\overline{g(z)}\, ds$. As usual, $H_2(D)$ stands for the Hardy–Szegö space. Here, $H_2(D)$ can be regarded as a linear subspace of $L_2(\partial D)$ consisting of those functions μ which satisfy

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\mu(z)}{z - \xi}\, dz = 0, \qquad \xi \in \mathbb{C} - \bar{D}.$$

Of course, $H_2(D)$ is itself a Hilbert space and every function $\mu \in H_2(D)$ determines exactly one analytic function in D defined by

$$f(\xi) = \frac{1}{2\pi i} \int_{\partial D} \frac{\mu(z)}{z - \xi}\, dz, \qquad \xi \in D. \tag{2.1}$$

μ is regarded as the boundary value of f (this refers to an arbitrary nontangential approach) and because of the above uniqueness, we shall not distinguish between $\mu(\xi)$ and $f(\xi)$, $\xi \in \partial D$.

In view of (2.1), point evaluations are bounded linear functionals on $H_2(D)$ and therefore $H_2(D)$ admist a reproducing kernel $\hat{K}(z, \bar{\xi})$ which is the classical Szegö kernel for D.

We introduce the notation

$$\mathscr{K}_1(z, \bar{\xi}) = \pi\tilde{K}(z, \bar{\xi}), \qquad \mathscr{K}_2(z, \bar{\xi}) = [2\pi\hat{K}(z, \bar{\xi})]^2, \qquad \mathscr{K}_3(z, \bar{\xi}) = \pi K(z, \bar{\xi}).$$

Clearly, the six functions

$$J_{ij}(z, \bar{\xi}) = \mathscr{K}_i(z, \bar{\xi})/\mathscr{K}_j(z, \bar{\xi}); \qquad i, j = 1, 2, 3; \quad i \neq j,$$

are conformally invariant and hence $J_{ij}(z, \bar{\xi}) \equiv 1$ when D is simply connected ($p = 1$). If D is not simply connected ($1 < p < \infty$) then a well-known chain of inequalities (cf. Hejhal [6, p. 107]) namely,

$$\mathscr{K}_1(z, \bar{z}) < \mathscr{K}_2(z, \bar{z}) < \mathscr{K}_3(z, \bar{z}), \qquad z \in D$$

holds. Consequently,

$$J_{13}(z, \bar{z}) < J_{23}(z, \bar{z}) < 1, \qquad J_{12}(z, \bar{z}) < 1, \qquad z \in D. \tag{2.2}$$

It is easily established that $\mathscr{K}_j(z, \bar{z})$ are monotonic with the domain D, that is

$$\mathscr{K}_j^{(D)}(z, \bar{z}) \leqslant \mathscr{K}_j^{(D_1)}(z, \bar{z}), \qquad z \in D_1 \subset D. \tag{2.3}$$

Using (2.3) one easily shows, by employing a method described in [1, p. 38], that

PROPOSITION 1. *Let $\xi \in \partial D$. Then within the angular sector* $|\arg(z - \xi)| \leqslant \alpha < \pi/2$, *it holds* $\lim_{z \to \xi} J_{ij}(z, \bar{z}) = 1$.

Since $J_{ij} = J_{ji}^{-1}$, $J_{ij} = J_{ij}(z) = J_{ij}(z, \bar{z})$, we consider only the three functions J_{ij}, $i < j$, $i, j = 1, 2, 3$. Clearly, $J_{ij} > 0$ and in view of (2.2) and Proposition 1 we obtain.

PROPOSITION 2. *If D is not simply connected* ($1 < p < \infty$), $J_{ij}(z)$, $i < j$, *assumes its minimum at an interior point of D.*

The usefulness of the functions $J_{ij}(z)$ is in the effective methods of their computation. In fact, since the above mentioned Hilbert spaces are separable each of them possesses a complete orthonormal basis, say $\{\phi_\nu^{(j)}(z)\}$; $j = 1, 2, 3$, $\nu = 1, 2, \ldots$. Then

$$K_j(z, \bar{\xi}) = \sum_{\nu=1}^{\infty} \phi_\nu^{(j)}(z) \overline{\phi_\nu^{(j)}(\xi)}, \qquad j = 1, 2, 3 \tag{2.4}$$

where

$$K_1(z, \bar{\xi}) = \tilde{K}(z, \bar{\xi}), \qquad K_2(z, \bar{\xi}) = \hat{K}(z, \bar{\xi}), \qquad K_3(z, \bar{\xi}) = K(z, \bar{\xi}).$$

The bilinear sum in (2.4) is independent of the choice of the orthonormal basis and it converges absolutely and uniformly on compacta of D.

3. DOUBLY CONNECTED DOMAINS

Let D be a doubly connected domain with no degenerate boundary component. Assume D has the modulus $1/r$ so that D can be represented as the annulus

$R = \{z: r < |z| < 1\}$, $0 < r < 1$. Here we have at our disposal the elliptic functions. The Bergman kernel is given by [1, p. 10]

$$K_3(z, \bar{\xi}) = \frac{1}{\pi z \bar{\xi}} \left\{ \mathscr{P}(\log z\bar{\xi}) + \frac{\eta_1}{\omega_1} - \frac{1}{2 \log r} \right\}. \tag{3.1}$$

Here $\mathscr{P}(u) = \mathscr{P}(u; \omega_1, \omega_2)$ is the Weierstrass $\mathscr{P}$-function with the half period $\omega_1 = \pi i$, $\omega_2 = \log r$ and $2\eta_j$ is the increment of the Weierstrassian ζ-functions with respect to the periods $2\omega_j$ ($j = 1, 2$). We note that, since the first period is $2\pi i$, the value of the $\mathscr{P}$-function does not depend on the value chosen for $\log z\bar{\xi}$. From (3.1) follows that the reduced Bergman kernel is

$$K_1(z, \bar{\xi}) = \frac{1}{\pi z \bar{\xi}} \{\mathscr{P}(\log z\bar{\xi}) + \eta_1/\omega_1\}. \tag{3.2}$$

Since the sequence $z^n/[2\pi(1 + r^{2n+1})]^{1/2}$, $n = \cdots, -1, 0, 1,\ldots$, forms an orthonormal basis for $H_2(R)$ the Szegö kernel for R is given by

$$K_2(z, \bar{\xi}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(z\bar{\xi})^n}{1 + r^{2n+1}}.$$

By a well-known formula of Garabedian [1, p. 119] we have

$$4\pi K_2^2(z, \bar{\xi}) = K_1(z, \bar{\xi}) + \alpha w'(z)\, \overline{w'(\xi)} \tag{3.3}$$

where $w(z) = \log |z|/\log r$ and so $w'(z) = z^{-1}/2 \log r$. Comparing the coefficients of $(z\bar{\xi})^{-1}$ in (3.3), and noting that the term with $(z\bar{\xi})^{-1}$ does not appear in $K_1(z, \bar{\xi})$ we obtain

$$\alpha = \frac{8 \log^2 r}{\pi} \sum_{n=0}^{\infty} \frac{r^{2n+1}}{(1 + r^{2n+1})^2}.$$

As usual, we set

$$\omega_3 = \omega_1 + \omega_2; \qquad e_j = \mathscr{P}(\omega_j), \quad j = 1, 2, 3,$$

so for $r = e^{\omega_2/\omega_1 \pi i}$, we have the identity (cf. Hancock [5, p. 483])

$$2\eta_1\omega_1 + 2e_3\omega_1^2 = 4\pi^2 \sum_{n=0}^{\infty} \frac{r^{2n+1}}{(1 + r^{2n+1})^2}.$$

Therefore,

$$\alpha = \frac{4 \log^2 r}{\pi^3} (\eta_1\omega_1 + e_3\omega_1^2).$$

This, together with (3.2), (3.3) and the Legendre relation

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}, \qquad \operatorname{Re}\left(\frac{\omega_2}{i\omega_1}\right) > 0,$$

implies that

$$K_2^2(z, \bar{\xi}) = \frac{1}{4\pi^2 z\bar{\xi}} \{\mathscr{P}(\log z\bar{\xi}) - e_3\}. \tag{3.4}$$

From (3.1), (3.2), (3.4) and the Legendre relation we obtain

$$\mathscr{K}_1(z, \bar{z}) = \frac{1}{|z|^2} \{\mathscr{P}(2 \log |z|) + a\}, \qquad a = \eta_1/\omega_1, \tag{3.5}$$

$$\mathscr{K}_2(z, \bar{z}) = \frac{1}{|z|^2} \{\mathscr{P}(2 \log |z|) - e_3\}, \tag{3.6}$$

$$\mathscr{K}_3(z, \bar{z}) = \frac{1}{|z|^2} \{\mathscr{P}(2 \log |z|) + b\}, \qquad b = \eta_2/\omega_2. \tag{3.7}$$

Incidentally, formula (3.6) settles a question raised in Kobayashi's book [7, p. 52]. Indeed, $\mathscr{K}_2(z, \bar{z}) \mid dz \mid^2$ is the Carathéodory metric for the annulus R.

The functions J_{ij}, $i < j$, are now given by

$$J_{12}(z) = 1 + \frac{a + e_3}{\mathscr{P} - e_3},$$

$$J_{13}(z) = 1 - \frac{b - a}{\mathscr{P} + b},$$

$$J_{23}(z) = 1 - \frac{b + e_3}{\mathscr{P} + b}, \qquad \mathscr{P} \equiv \mathscr{P}(2 \log |z|).$$

Here,, of course $J_{ij}(z) < 1$, $J_{ij}(r) = J_{ij}(1) = 1$ and $J_{ij}(z) = J_{ij}(|z|) = J_{ij}(r/|z|)$. Consequently $J_{ij}(z)$ has extremal points on all the circle $|z| = r^{1/2}$. We now come to the main theorem.

THEOREM 1. *For $\rho = |z|$, $J_{ij}(\rho)$, $i < j$, has only one extremal point in $(r, 1)$.*

In view of Proposition 1 and the above remark this extremal point is at $r^{1/2}$ and it is a minimum point.

Proof. By a direct computation

$$\frac{d}{d\rho} J_{12}(\rho) = -\frac{2}{\rho} (a + e_3) (\mathscr{P} - e_3)^{-2} \mathscr{P}'(2 \log \rho).$$

This is equal to zero if and only if $\mathscr{P}'(2 \log \rho) = 0$. However,

$$[\mathscr{P}'(u)]^2 = 4(\mathscr{P} - e_1)(\mathscr{P} - e_2)(\mathscr{P} - e_3), \qquad \mathscr{P} = \mathscr{P}(u),$$

and this equals zero if and only if $u = 2 \log \rho = \omega_2 = \log r$ or if and only if $\rho = r^{1/2}$. The proof for J_{13} and J_{23} is of course similar.

This theorem yields at once an effective method for determining the modulus $1/r$ of a given doubly connected domain D. Indeed, $J_{ij}^{(D)}(z)$ can be computed via (2.4) and by its comformal invariance and Theorem 1 we have the implicit equation in r

$$J_{ij}(r^{1/2}) = \min_{z \in D} J_{ij}^{(D)}(z), \qquad i < j. \tag{3.8}$$

The unique solution r of (3.8) is the reciprocal of the modulus of D. We of course exhibited 3 methods for determining r and it seems that the choice of $i = 1, j = 2$ is the most efficient one.

4. The Numerical Algorithm

Let D be a doubly connected domain bounded by two disjoint Jordan curves Γ_0 and Γ_1 and suppose, without any loss of generality, that the origin lies in the "hole" of D (i.e., $0 \in \operatorname{Int} \Gamma_1$). In this case, $\{z^n\}_{n=-\infty}^{\infty}$ forms a complete set in $L_2H(D)$ and in $H_2(D)$ while $\{z^n\}_{n=-\infty}^{\infty}$, $n \neq -1$, forms a complete set in $L_2{}^sH(D)$. We compute the moments of these sequences with respect to the corresponding scalar products and then employ the Gram–Schmidt procedure to obtain the orthonormal bases. From these we compute the kernel functions and form the $J_{ij}^{(D)}(z)$, $i < j$. Finally, we set the equation (3.8) and solve for r. The actual numerical computation is executed exactly as in [3]. We also remark that $J_{ij}(r^{1/2})$ in (3.8) can be easily calculated numerically. For example

$$J_{12}(r^{1/2}) = 1 - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{r^n}{1 + r^{2n+1}}\right)^2 \Big/ \left(\sum_{n=0}^{\infty} \frac{r^n}{1 + r^{2n+1}}\right)^2$$

and similar expression, but involving $\log r$, holds for $J_{13}(r^{1/2})$ and for $J_{23}(r^{1/2})$.

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